One-loop effective brane action

Tae-Hun Lee

Department of Physics, Purdue University,
525 Northwestern Avenue, West Lafayette, IN 47907-2036, U.S.A.
E-mail: `lee109@physics.purdue.edu`

ABSTRACT: The one-loop effective action for a p brane embedded in a $D = p+2$ Minkowski spacetime in the static gauge is calculated. Rescaling the quantum fluctuation by $\sqrt{-g_0}$ evaluated on the background brane leads to the one-loop effective action expressed only in terms of infrared and ultraviolet divergent geometric scalars. After the infrared divergences are absorbed into the quantum fluctuation, there remains the finite number of ultraviolet divergences. This implies that the $D = p+2$ Poincaré symmetry and the $D = p+1$ general coordinate invariance are preserved in one-loop order.

KEYWORDS: Gauge Symmetry, Space-Time Symmetries, Brane Dynamics in Gauge Theories, p-branes.

Contents

1. Introduction

Although quantization of a bosonic string allows only a special dimension, $D = 26$ [1], brane world scenarios in other dimensions are still being actively studied in the context of long wavelength brane oscillation effective actions [2, 3]. On the other hand, there could be many possible classically equivalent descriptions of branes which may have different quantum aspects [4]. For example, the Polyakov action [1] is often a useful choice because of its rich insight to the physics and convenience due to its covariant nature. Despite the presence of the square root the Nambu-Goto action [5] is still widely used for fundamental theories as well as description of specific models, for example, numerical analysis of gauge interactions, especially, the strong interaction [6]. Especially, in this paper, to see its effective quantum structures of branes in various dimensions the Nambu-Goto action is used to describe a brane embedded in a certain target dimension.

The Nambu-Goto action is an invariant world volume element constructed from the induced metric, so it has the higher dimensional Poincaré symmetry as well as the reparametrization invariance. Because of the reparametrization invariance symmetry the gauge fixing is required. In general, the gauge fixing breaks both of the symmetries. However, the higher dimensional Poincaré symmetry can be realized by the nonlinear transformation among parameters and the fields [2, 3]. It is a question whether this nonlinear

symmetry is preserved in every order of loops. The purpose of this paper is to calculate the one-loop effective action for a p brane embedded in a $D = p + 2$ Minkowski spacetime in the static gauge [2, 3], focusing on the symmetries.

Specifically, the action for a p brane embedded in a $D = p + 2$ Minkowski spacetime can be described by a single scalar field in the static gauge, where the ghost contribution is absent as it is in the axial gauge in non-Abelian gauge theory. In a naive expansion of the action around the classical field, the term corresponding to one-loop order appears as if it were a scalar field action in the gravitational field, denoted by the 0 subscript, as the classical non-tensorial object $g_0^{-2/(p-1)} g_{0mn}$, where $g_0 \equiv - \det g_{0mn}$. Hence, the classical symmetries of the effective action are not obviously preserved. One methodology to attempt to maintain the symmetry structure is to return to the original action and expand the action about classical fields without fixing the gauge. The Faddeev-Popov gauge fixing and ghost terms must be added to the action. It turns out that it is difficult to integrate the generating functional to get the determinant of the double index non-symmetric metric where the $D = p + 2$ and $D = p + 1$ Lorentz indices are entangled. The geodesic expansion [7, 8] could be another possible way though it was not considered in this paper.

This paper shows that the effective action in one-loop order can be expressed in terms of geometric scalars by rescaling the quantum fluctuation by its own classical Lagrangian density $\sqrt{-g_0}$. The path integral measure is not changed by this local scale change in the dimensional regularization, contrasted to the canonical, Hamiltonian formalism [9]. Therefore, a ghost action due to the Jacobian factor from the local scale change does not appear. But the origin of this scale transformation and the complete understanding of the consequent divergent structure still remain a question to be addressed.

2. The Nambu-Goto action in the static gauge

The symmetries of the brane action are made manifest in one-loop in this section. It will be shown that this can be done by rescaling the quantum fluctuation. The action for a p brane embedded in a $D = p + 2$ Minkowski spacetime is given as the Nambu-Goto action [5].

$$S = -\sigma \int d^{p+1}x \sqrt{- \det[\eta_{\mu\nu} \partial_m X^\mu(x) \partial_n X^\nu(x)]}, \qquad (2.1)$$

where $\mu, \nu = 0, 1, 2, \ldots, p + 1$ and $m, n = 0, 1, 2, \ldots, p$. It is well known that it has the $D = p + 2$ Poincaré symmetry of the bulk as well as the reparametrization invariance, or equivalently, the gauge symmetry on the scalar fields X^μs, i.e., $X^\mu \to X^\mu + \omega^a \partial_a X^\mu$ [11]. The static gauge allows one to fix the gauge by simply choosing the parameters x^m equal to the fields X^m. In the path integral method, the gauge fixing condition $f^m = X^m - x^m$ yields the Jacobian in the functional integration measure, $\det[\frac{\delta f^m(x)}{\delta \omega^n(y)}] = \det[\delta(x-y)\partial_n X^m]$, which becomes independent of X^m when combined with $\delta[f^m - \chi^m]$, where χ^m is an arbitrary function to be integrated out in the path integral. Therefore, there is no ghost contribution in the static gauge as in the axial gauge in non-Abelian gauge theory. With the only remaining degree of freedom the $(p + 2)$th coordinate, $\phi \equiv X^{p+1}$, the Lagrangian

density becomes [2, 3]

$$\mathscr{L} = -\sigma\sqrt{1 - \partial_m\phi\partial^m\phi} = -\sigma\sqrt{-\det g_{mn}} \equiv -\sigma g^{\frac{1}{2}}, \tag{2.2}$$

where $\partial^m \equiv \eta^{mn}\partial_n$, $g_{mn} = \eta_{mn} - \partial_m\phi\partial_n\phi$ and its inverse $g^{mn} = \eta^{mn} + g^{-1}\partial^m\phi\partial^n\phi$. The effective action Γ is defined by,

$$e^{\frac{i}{\hbar}\Gamma[\phi_0,J]} = \int \mathscr{D}\phi \exp\frac{i}{\hbar}\int d^{p+1}x[\mathscr{L}(\phi) + (\phi - \phi_0)J] = \int \mathscr{D}\varphi \exp\frac{i}{\hbar}\int d^{p+1}x[\mathscr{L}(\varphi + \phi_0) + \varphi J], \tag{2.3}$$

Expanding the shifted action about a classical field ϕ_0 and keeping it up to the second order in the fluctuation φ as $\phi = \varphi + \phi_0$,

$$\int d^{p+1}x\mathscr{L}(\varphi + \phi_0) = -\sigma\int d^{p+1}xg_0^{\frac{1}{2}} + \sigma\int d^{p+1}xg_0^{-\frac{1}{2}}\partial^m\phi_0\partial_m\varphi$$

$$+\frac{1}{2}\sigma\int d^{p+1}xg_0^{-\frac{1}{2}}g_0^{mn}\partial_m\varphi\partial_n\varphi + \cdots$$

$$= \sigma\int d^{p+1}xg_0^{\frac{1}{2}}\left(-1 + \frac{1}{2}g_0^{-1}g_0^{mn}\partial_m\varphi\partial_n\varphi + \cdots\right), \tag{2.4}$$

where the linear term has been eliminated by the classical equation of motion $\partial^m(g_0^{-\frac{1}{2}}\partial_m\phi_0) = 0$. The quadratic term corresponding to one-loop order looks like the scalar field action in curved spacetime with the metric $\tilde{g}_{0mn}$. By taking a determinant of the background part

$$g_0^{-\frac{1}{2}}g_0^{mn} = \tilde{g}_0^{\frac{1}{2}}\tilde{g}_0^{mn}, \tag{2.5}$$

the metric for such a gravitational field $\tilde{g}_{0mn}$ can be identified with

$$\tilde{g}_{0mn} = g_0^{-2/(p-1)}g_{0mn}. \tag{2.6}$$

However, $g_0^s g_{0mn}(s \neq 0)$ does not transform like a tensor under the $D = p + 2$ Poincaré symmetry [2, 3] (For the details see eqs. (2.17), (2.18), (2.19) in section 2.1). In the end, the effective action would not be expected to be manifestly invariant under these symmetries. In order to allow for a manifestly invariant action a classical field dependent background rescaling of the fluctuation is utilized.

$$\varphi \to \alpha\varphi, \tag{2.7}$$

where $\alpha = \alpha(\phi_0)$ is a local function of ϕ_0. This scale transformation changes the functional measure by an infinite constant or $\delta(0) \times$ constant, in which $\delta(0)$ vanishes in the dimensional regularization.

$$\mathscr{D}\varphi \to \mathscr{D}\varphi \det[\alpha] = \mathscr{D}\varphi e^{\delta(0)\int d^{p+1}x\ln\alpha} = \mathscr{D}\varphi. \tag{2.8}$$

The scale change produces φ^2 term in the action.

$$\int d^{p+1}xg_0^{-\frac{1}{2}}g_0^{mn}\partial_m(\alpha\varphi)\partial_n(\alpha\varphi) = \int d^{p+1}xg_0^{-\frac{1}{2}}g_0^{mn}\alpha^2\partial_m\varphi\partial_n\varphi \tag{2.9}$$

$$+\int d^{p+1}x\left[g_0^{-\frac{1}{2}}g_0^{mn}\partial_m\alpha\partial_n\alpha - \partial_n\left(g_0^{-\frac{1}{2}}g_0^{mn}\frac{1}{2}\partial_m\alpha^2\right)\right]\varphi^2.$$

If α chosen to be $g_0^{\frac{1}{2}}$, the first term is obviously in a manifestly invariant form. Moreover, as explicitly shown in the appendix B.4, using the equation of motion the background parts in the second term reduce to

$$g_0^{-\frac{1}{2}} g_0^{mn} \partial_m g_0^{\frac{1}{2}} \partial_n g_0^{\frac{1}{2}} - \partial_n \left[g_0^{-\frac{1}{2}} g_0^{mn} \frac{1}{2} \partial_m \left(g_0^{\frac{1}{2}} \right)^2 \right] = g_0^{\frac{1}{2}} R_0, \qquad (2.10)$$

where R_0 is the Ricci scalar based on g_{0mn}. Thus, it is obtained that $D = p + 2$ Poincaré invariant world volume action for the brane is secured in one-loop order. (The detailed explanation is provided in the next section.) As a result of the rescaling, the effective action in one-loop at $J = 0$ is

$$\Gamma[\phi_0]_{\text{one-loop}} = -i\hbar \ln \int \mathscr{D}\varphi \exp \frac{i}{\hbar} \sigma \int d^{p+1}x \frac{1}{2} g_0^{\frac{1}{2}} (g_0^{mn} \partial_m \varphi \partial_n \varphi + R_0 \varphi^2). \qquad (2.11)$$

It can be immediately recognizable that except for σ this is the same as the generating function of the connected Green function for a massless scalar field in background gravity, $W[0]$ in $D = p + 1$ with $\xi = -1$ [10].

$$W[0] = -i\hbar \ln \int \mathscr{D}\phi \exp \frac{i}{\hbar} \int d^{p+1}x \frac{1}{2} \sqrt{-g} [g^{mn} \partial_m \phi \partial_n \phi - (m^2 + \xi R)\phi^2]. \qquad (2.12)$$

2.1 $D = p + 2$ Poincaré symmetry

This section is intended to explicitly show that the metric g_{0mn} and its inverse g_0^{mn} can be treated as a tensor on the $D = p + 2$ Poincaré symmetry and consequently any geometrical tensor made of these metrics and derivatives is also such a tensor. (Note that the indices in (x_m, b_m, v_m) are raised by η^{mn}, for a shorthand notation 0 subscript is dropped and the following notations are used. $\partial_m \phi \equiv v_m$, $b \cdot v \equiv \eta^{mn} b_m v_n$, $v^2 \equiv v_m v^m$, etc.)

In the static gauge the $D = p + 2$ Poincaré symmetry is realized on the coordinates and field, [2, 3], as

$$x'^m = x^m + a^m - \phi b^m + \epsilon^{mnr} \alpha_n x_r,$$
$$\Delta \phi = z - b_m x^m, \qquad (2.13)$$

where Δ is a total variation and z and a^m are a broken and an unbroken infinitesimal translational transformation parameter, respectively while b_m and α_n are a broken and an unbroken infinitesimal Lorentz transformation parameter, respectively. Since the action still has the $D = p + 1$ unbroken Poincaré symmetry, the only parts to be considered in eq. (2.13) are the $D = p + 2$ higher dimensional broken symmetry transformations. Note that the constant z can be ignored since the action depends on only derivatives of ϕ.

$$x'^m = x^m - \phi b^m,$$
$$\Delta \phi = -b_m x^m. \qquad (2.14)$$

If g_{mn} and g^{mn} transform under the given transformations as tensors $g'_{mn} = \frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}g_{pq}$ and $g'^{mn} = \frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^q}g^{pq}$, technically they can be treated as a tensor.

$$
\begin{aligned}
g'_{mn} &= \frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}g_{pq} \\
&= (\delta_n^p + b^p v_n)(\delta_m^q + b^q v_m)g_{pq} \\
&= \eta_{mn} - v_m v_n - b_m v_n - b_n v_m + 2(b\cdot v)v_m v_n \\
&= g_{mn} - b_m v_n - b_n v_m + 2(b\cdot v)v_m v_n, \\
g'^{mn} &= \frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^q}g^{pq} \\
&= (\delta_p^n - b^n v_p)(\delta_q^m - b^m v_q)g^{pq} \\
&= \eta^{mn} - b^n v^m - b^m v^n + g^{-1}v^m v^n - g^{-1}v^m b^n v^2 - g^{-1}v^n b^m v^2 \\
&= g^{mn} - g^{-1}b^m v^n - g^{-1}b^n v^m.
\end{aligned}
\tag{2.15}
$$

By using the transformations $v'_m = v_m + v_m(b\cdot v) - b_m$ and $g' = g + 2g(b\cdot v)$ and their definitions $g_{mn} = \eta_{mn} - v_m v_n$ and $g^{mn} = \eta^{mn} + g^{-1}v^m v^n$, it can be checked that they indeed follow the same transformations.

$$
\begin{aligned}
g'_{mn} &= g_{mn} + \delta[\eta'_{mn}] + \delta[v_m]v_n + v_m\delta[v_n] \\
&= g_{mn} + [v_m(b\cdot v) - b_m]v_n + v_m[v_n(b\cdot v) - b_n] \\
&= g_{mn} - b_m v_n - b_n v_m + 2v_m v_n(b\cdot v), \\
g'^{mn} &= g^{mn} + (-g^{-2})(2gb\cdot v)v^m v^n + g^{-1}\delta[v^m]v^n + g^{-1}v^m\delta[v^n] \\
&= g^{mn} - 2g^{-1}b\cdot v v^m v^n + g^{-1}(2v^m v^n b\cdot v - b^m v^n - b^n v^m) \\
&= g^{mn} - g^{-1}b^m v^n - g^{-1}b^n v^m.
\end{aligned}
\tag{2.16}
$$

Therefore, any geometric tensor based on these metric tensors transforms like a tensor under the given transformations. Based on these transformations, it can be seen that $g^s g_{mn}(s\neq 0)$ does not transform as a tensor since g is not a scalar. Explicitly,

$$
\Delta g = -2v^m\Delta[v_m] = -2(v_m b\cdot v - b_m)v^m = 2g(b\cdot v) \tag{2.17}
$$

and thus

$$
\begin{aligned}
[g^s g_{mn}]' &= g^s g_{mn} + sg^{s-1}\Delta g g_{mn} + g^s\Delta g_{mn} \\
&= g^s g_{mn} + sg^{s-1}2g(b\cdot v)g_{mn} + g^s[-b_m v_n - b_n v_n + 2v_m v_n(b\cdot v)] \\
&= g^s[g_{mn}(1 - 2s(b\cdot v)) - b_m v_n - b_n v_n + 2v_m v_n(b\cdot v)].
\end{aligned}
\tag{2.18}
$$

This is not a tensor transformation, that is,

$$
[g^s g_{mn}]' \neq \frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}g^s g_{pq} = g^s[g_{mn} - b_m v_n - b_n v_m + 2(b\cdot v)v_m v_n]. \tag{2.19}
$$

Hence, $g_0^{-2/(p-1)}g_{0mn}$ in the action is not a tensor.

2.2 Scalar field in background gravity

In the previous sections, it has been seen that the one-loop effective action is given as $W[J = 0]$ in a scalar field theory in background gravity. Here, the result of calculation of $W[J = 0]$ is introduced from [Birrell & Davies, (1982)][10]. Suppose a scalar field is in a background curved spacetime described by the Lagrangian density

$$\mathscr{L} = \frac{1}{2}\sqrt{-g}[g^{mn}\partial_m\phi\partial_n\phi - \left(m^2 + \xi R\right)\phi^2],$$ (2.20)

where ξ is a numerical factor, m is the mass of the field ϕ and R is the Ricci scalar. The generating functional Z and the generating function of the connected Green functions W in $D = p + 1$ are

$$Z[J] = \int \mathscr{D}\phi \exp\left[\frac{i}{\hbar}\int d^{p+1}x(\mathscr{L} + J\phi)\right], \quad W = -i\hbar\ln Z[0].$$ (2.21)

In the dimensional regularization, W is given in eq. (6.41), p.159, Birrell & Davies, (1982) [10].

$$W = \lim_{n\to p+1}\hbar\int d^nx\sqrt{-g}\frac{1}{2}(4\pi)^{-n/2}\sum_{j=0}^{\infty}a_j(x)\int_0^{\infty}(is)^{j-1-n/2}e^{-im^2s}ids$$

$$= \lim_{n\to p+1}\hbar\int d^nx\sqrt{-g}\frac{1}{2}(4\pi)^{-n/2}\sum_{j=0}^{\infty}a_j(x)(m^2)^{n/2-j}\Gamma(j - n/2),$$ (2.22)

where

$$\begin{aligned}
a_0(x) &= 1, \\
a_1(x) &= \left(\frac{1}{6} - \xi\right)R, \\
a_2(x) &= \frac{1}{180}R_{abcd}R^{abcd} - \frac{1}{180}R^{ab}R_{ab} - \frac{1}{6}\left(\frac{1}{5} - \xi\right)\Box R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2, \\
&\cdots .
\end{aligned}$$ (2.23)

It is necessary to recognize that $a_j(x)$ is a $2j-$derivative geometric scalar with respect to the metric. This implies that the net remaining number of g^{mn} after contraction with g_{mn} in $a_j(x)$ is j.

2.3 One-loop effective action

Now the above result eq. (2.22) can be identified with the one-loop effective brane action when the mass parameter is sent to zero. This limit produces an infinite series of divergences due to $(m^2)^{n/2-j}$ when $n/2 - j < 0$. Regardless of this limit, there is another source of divergence from $\Gamma(j - n/2)$ when $n/2 - j \geq 0$. One can think that in eq. (2.22) e^{-im^2s} is introduced just for regulating the first divergences in the DeWitt-Schwinger representation of a massless scalar field action. The origin of these divergences comes from $a_j(x)\int d^nke^{-iky}(k^2 - m^2)^{-j-1}$ in the propagator expansion, together with the extra power factor m^2 from the final effective action $W[0] = -\frac{1}{2}i\text{tr}[\ln(-G_F)]$ when $n/2 - j < 0$ [10],

for which zero momentum mode gives a divergence in the massless limit. Similarly, the divergences in $\Gamma(j - n/2)$ originate from the other limit $k \to \infty$ when $n/2 - j \geq 0$. In this reason, they can be called infrared and ultraviolet divergences, respectively. (For the detailed power counting of the mass parameter, see eq. (3.130), p.74 for the propagator expression and also eq. (6.34), p157 in Birrell & Davies, (1982)) [10].

The infrared divergences $(m^2)^{n/2-j}$ in eq. (2.22) seem to be somewhat fictitious. A constant scale change of φ causes only adding number in the effective action eq. (2.11). It is equivalent to the corresponding scale change in g_{mn}. Therefore, if this scale is properly chosen to cancel $(m^2)^{n/2-j}$ factor, i.e.,

$$\varphi \to (m^2)^{-n/4+1/2}\varphi, \tag{2.24}$$

equivalently,

$$g_{0mn} \to (m^2)^{-1}g_{0mn} \tag{2.25}$$

and hence

$$a_j \to (m^2)^j a_j. \tag{2.26}$$

They bring the scale $(m^2)^{-n/2+j}$ in the one-loop effective action eq. (2.11). As a result, only ultraviolet divergences remain in the one-loop effective action.

$$\Gamma[\phi_0]_{\text{one-loop}} = \lim_{n \to p+1} \frac{\hbar}{2}(4\pi)^{-n/2}\sigma^{n/(n-2)} \int d^n x g_0^{\frac{1}{2}} \sum_{j=0}^{\infty} \sigma^{-2j/(n-2)} a_{j0}(x)\Gamma(j - n/2) \tag{2.27}$$

The terms with $j > n/2$ in the effective action are finite. Therefore, there remains only the finite number of ultraviolet divergences, which can be renormalized by adding counter terms. In addition, these results can be always simplified further by using the constraint eq. (B.8) from the equation of motion. (For the derivation, refer to appendix B.3)

$$R_0^{abcd} R_{0abcd} = 2(R_0^2 - R_0^{ab} R_{0ab}). \tag{2.28}$$

where note that in $n = 2$ case this relation is just a general identity, not a constraint.

2.4 Effective correction to Einstein's field equation

Since the quantum corrections to the classical brane action have the $D = p + 1$ general coordinate invariance as well as the $D = p + 2$ Poincaré symmetry, they can be effectively treated as gravitational interaction with the metric g_{0mn}. If the higher order extra corrections in the effective action can be considered a new effective contribution to a gravitational action, the effective classical field equation can be constructed with modification by these terms. For example, for $p + 1 = 4$ case [10], as the infinite terms including a_0, a_1 and a_2 require renormalization of the cosmological constant Λ, the gravitational constant G from R and new couplings from $\frac{1}{180}R_{abcd}R^{abcd} - \frac{1}{180}R^{ab}R_{ab} - \frac{1}{6}(\frac{1}{5} - \xi)\Box R$, respectively, the last higher order contribution effectively modifies the field equation.

$$R_{0mn} - \frac{1}{2}R_0 g_{0mn} + \Lambda g_{0mn} + aH_{0mn}^{(1)} + bH_{0mn}^{(2)} + cH_{0mn} = 0, \tag{2.29}$$

where a, b and c are the corresponding renormalized coefficients after the divergent coefficients are absorbed into their bare couplings and the higher derivative tensors [10] are

$$
\begin{aligned}
H_{0mn}^{(1)} &\equiv g_0^{-\frac{1}{2}} \frac{\delta}{\delta g_0^{mn}} \int d^4x\, g_0^{\frac{1}{2}} R_0^2 \\
&= 2R_{0;mn} - 2g_{0mn}\Box R_0 - \frac{1}{2}g_{0mn}R_0^2 + 2R_0 R_{0mn}, \\
H_{0mn}^{(2)} &\equiv g_0^{-\frac{1}{2}} \frac{\delta}{\delta g_0^{mn}} \int d^4x\, g_0^{\frac{1}{2}} R_{0ab} R_0^{ab} \\
&= 2R_{0m;na}^a - \Box R_{0mn} - \frac{1}{2}g_{0mn}\Box R_0 + 2R_{0m}^a R_{0an} - \frac{1}{2}g_{0mn}R_0^{ab}R_{0ab}, \qquad (2.30) \\
H_{0mn} &\equiv g_0^{-\frac{1}{2}} \frac{\delta}{\delta g_0^{mn}} \int d^4x\, g_0^{\frac{1}{2}} R_{0abcd} R_0^{abcd} \\
&= -\frac{1}{2}g_{0mn}R_0^{abcd}R_{0abcd} + 2R_{0mabc}R_{0n}^{abc} - 4\Box R_{0mn} + 2R_{0;mn} - 4R_{0ma}R_{0n}^a \\
&\quad + 4R_0^{ab}R_{0ambn},
\end{aligned}
$$

where note that the term $\Box R_0$ in a_2 has been ignored because $g_0^{\frac{1}{2}}\Box R_0$ is a total derivative and thus $\int d^4x\, g_0^{\frac{1}{2}}\Box R_0 = 0$ and for simplification $H_{mn} = -H_{mn}^{(1)} + 4H_{mn}^{(2)}$ may be applied, obtained from the well-known identity in $D = 4$, $g^{-\frac{1}{2}}\frac{\delta}{\delta g^{mn}}\int d^4x(R_{abcd}R^{abcd} + R^2 - 4R_{ab}R^{ab}) = 0$ [10].

3. Conclusion

The action for a p brane embedded in a $D = p+2$ Minkowski spacetime has been examined in one-loop order. It was found that the special local scale transformation enables the $D = p+1$ general coordinate invariance and the $D = p+2$ Poincaré symmetry to be preserved in classical geometric form in one-loop. These geometric scalars include ultraviolet divergences and an infinite series of infrared divergences. However, an infinite series of the infrared divergences can be removed by the global scale change.

A. Notations

Greek $(\mu, \nu, \ldots)$ and Latin letters $(m, n, \ldots)$ are used for $D = p+2$ and $D = p+1$ Lorentz indices, respectively. The metric convention $\eta_{mn} = (1, -1, -1, -1, \ldots)$ is used. Also, the following notations are introduced for simplicity.

$$
\begin{aligned}
&v_m \equiv \partial_m\phi, \quad v_{mn} \equiv \partial_m\partial_n\phi, \qquad \partial_{mn} \equiv \partial_m\partial_n, \quad v^m \equiv \eta^{mn}v_n, \quad v^2 = v_m v^m, \quad g \equiv \det(-g_{mn}), \\
&p_{,\alpha} \equiv \partial_\alpha p, \qquad p_{;\beta}^\alpha \equiv p_{,\beta}^\alpha + \Gamma_{\mu\beta}^\alpha p^\mu, \quad p_{\alpha;\beta} \equiv p_{\alpha,\beta} - \Gamma_{\alpha\beta}^\mu p_\mu.
\end{aligned}
$$

For instance, $g_{mn} = \eta_{mn} - v_m v_n$ and $g^{mn} = \eta^{mn} + g^{-1}v^m v^n$.

B. Curvature tensors and scale transformation

This section provides explicit derivations of all the necessary geometric scalars necessary for eq. (2.24) and eq. (2.10) with the metric $g_{mn} = \eta_{mn} - v_m v_n$.

B.1 Christoffel symbol

The corresponding Christoffel symbol is

$$
\begin{aligned}
\Gamma^a_{bc} &= \frac{1}{2}g^{am}(g_{mb,c} + g_{mc,b} - g_{bc,m}) \\
&= \frac{1}{2}\left(\eta^{am} + \frac{1}{g}v^a v^m\right)(-2v_m v_{bc}) \\
&= -g^{-1}v^a v_{bc}.
\end{aligned}
\tag{B.1}
$$

B.2 Extrinsic curvature

Since all the geometric scalars can be expressed with the extrinsic curvature and it is also a tensor under the higher dimensional Poincaré symmetry, it may be convenient to introduce it. The extrinsic curvature is defined as follows.

$$
K_{ab} = n_{\alpha;\beta}e^\alpha_a e^\beta_b,
\tag{B.2}
$$

where the unit normal vector $n_\alpha = \frac{\epsilon\Phi_{,\alpha}}{\sqrt{|g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}|}}$ to the surface $\Phi(X) = 0$ and $\epsilon = \pm 1$ (+ for a timelike surface, − for a spacelike surface). For a p brane hypersurface embedded in a $D = p + 2$ Minkowski spacetime, $\Phi(X) = \phi(x) - X^{p+1} = 0$ where $\phi(x)$ is an arbitrary function of x,

$$
n_a = \frac{\epsilon v_a}{g^{\frac{1}{2}}},\ n_{p+1} = -\frac{\epsilon}{g^{\frac{1}{2}}},\ e^i_a = \delta^i_a,\ e^{p+1}_a = v_a.
\tag{B.3}
$$

Using the only nonzero components, $n_{p+1;m} = -\frac{\epsilon}{2}g^{-\frac{3}{2}}\partial_m v^2$ and $n_{m;n} = \epsilon g^{-\frac{1}{2}}v_{mn} + \frac{\epsilon}{2}g^{-\frac{3}{2}}\partial_n v^2 v_m$, the extrinsic curvature is found.

$$
K_{mn} = g^{-\frac{1}{2}}v_{mn},
\tag{B.4}
$$

where ϵ is chosen to be 1. Note this is a symmetric tensor. When the scalar $K_{mn}g^{mn}$ is made of the classical field, it vanishes by the equation of motion.

$$
K_{0mn}g^{0mn} = g_0^{-\frac{1}{2}}v_{0mn}(\eta^{mn} + g_0^{-1}v_0^m v_0^n) = g_0^{-\frac{1}{2}}\left(\partial \cdot v_0 + \frac{1}{2g_0}v_0 \cdot \partial v_0^2\right) = 0.
\tag{B.5}
$$

B.3 Curvature tensors

All the curvature tensors are functions of the extrinsic curvature tensors and the metric.

$$
\begin{aligned}
R_{abcd} &= \frac{1}{g}(v_{ad}v_{bc} - v_{bd}v_{ac}) = K_{ad}K_{bc} - K_{bd}K_{ac}, \\
R^{abcd} &= K^{ad}K^{bc} - K^{bd}K^{ac}, \\
R_{ab} &= g^{mn}R_{manb} = g^{mn}(K_{mb}K_{na} - K_{mn}K_{ab}) = K_{am}K^m_b - K_{ab}K, \\
R^{ab} &= g^{am}g^{bn}R_{mn} = g^{am}g^{bn}(K_{mp}K^p_n - K_{mn}K) = K^a_p K^{bp} - K^{ab}K,
\end{aligned}
\tag{B.6}
$$

where $K \equiv g^{ab}K_{ab}$, $K^a_b \equiv g^{am}K_{mb}$ and $K^{ab} \equiv g^{am}g^{bn}K_{mn}$. All the curvature scalars are simply expressed in terms of the contractions of extrinsic curvature tensors.

$$
\begin{aligned}
R &= g^{ab}(K_{am}K^m_b - K_{ab}K) = K^a_b K^b_a - K^2, \\
R^{ab}R_{ab} &= (K^a_n K^{bn} - K^{ab}K)(K_{am}K^m_b - K_{ab}K) \\
&= K^a_n K^n_b K^b_m K^m_a - 2K^a_m K^m_b K^b_a K + K^a_b K^b_a K^2, \\
R^{abcd}R_{abcd} &= 2[(K^a_b K^b_a)^2 - K^a_b K^b_c K^c_d K^d_a].
\end{aligned}
\tag{B.7}
$$

With $K_0 = 0$, the identity can be easily obtained.

$$R_0^{abcd} R_{0abcd} = 2(R_0^2 - R_0^{ab} R_{0ab}).$$ (B.8)

The tensors needed to calculate the four-derivative curvature scalars are

$$K_b^a = \left(\eta^{am} + \frac{1}{g} v^a v^m\right) g^{-\frac{1}{2}} v_{mb} = g^{-\frac{1}{2}} v_b^a + v^a \partial_b g^{-\frac{1}{2}} = \partial_b \left(g^{-\frac{1}{2}} v^a\right),$$

$$K_b^m K_{ma} = g^{-\frac{1}{2}} v_{mb} \left(g^{-\frac{1}{2}} v_a^m + \frac{1}{2} g^{-\frac{3}{2}} v^m \partial_a v^2\right) = g^{-1} \left(v_{mb} v_a^m + \frac{1}{4g} \partial_a v^2 \partial_b v^2\right),$$

$$K_a^m K_m^b = g^{-3} \left[g^2 v_a^m v_m^b + \frac{g}{4} \partial_a v^2 \partial^b v^2 + \frac{g}{2} v^b v_a^m \partial_m v^2 + \frac{1}{4}(v \cdot \partial v^2) v^b \partial_a v^2\right],$$

$$K_a^m K_m^b K_b^c = g^{-\frac{9}{2}} \left[g^3 v_a^m v_m^b v_b^c + \frac{g^2}{4} \partial_a v^2 \partial^b v^2 v_b^c + \frac{g^2}{4} \partial^c v^2 v_a^m \partial_m v^2 + \frac{g}{8}(v \cdot \partial v^2) \partial^c v^2 \partial_a v^2\right]$$

$$+ g^{-\frac{9}{2}} \left[\frac{g^2}{2} v^c v_a^m v_m^b \partial_b v^2 + \frac{g}{8} v^c \partial_a v^2 \partial v^2 \cdot \partial v^2 + \frac{g}{4} v^c (v \cdot \partial v^2) v_a^m \partial_m v^2\right.$$

$$\left. + \frac{1}{8} v^c (v \cdot \partial v^2)^2 \partial_a v^2\right].$$ (B.9)

Then, the scalars for the four-derivative curvature scalars are constructed from them.

$$K_a^m K_m^a = g^{-3} \left[g^2 v_a^m v_m^a + \frac{g}{2} \partial v^2 \cdot \partial v^2 + \frac{1}{4}(v \cdot \partial v^2)^2\right],$$

$$K_a^m K_m^b K_b^a = g^{-\frac{9}{2}} \left[g^3 v_a^m v_m^b v_b^a + \frac{3g^2}{4} \partial_a v^2 \partial^b v^2 v_b^a + \frac{3g}{8}(v \cdot \partial v^2)(\partial v^2 \cdot \partial v^2) + \frac{1}{8}(v \cdot \partial v^2)^3\right],$$

$$K_a^m K_m^b K_b^c K_c^a = g^{-6} \left[g^4 v_a^m v_m^b v_c^c v_c^a + g^3 \partial_a v^2 \partial^b v^2 v_b^c v_c^a + \frac{g^2}{2}(v \cdot \partial v^2) \partial^c v^2 \partial_a v^2 v_c^a\right.$$ (B.10)

$$\left. + \frac{g^2}{8}(\partial v^2 \cdot \partial v^2)^2 + \frac{g}{4}(v \cdot \partial v^2)^2(\partial v^2 \cdot \partial v^2) + \frac{1}{16}(v \cdot \partial v^2)^4\right].$$

B.4 Scale transformation

Note that for simplicity 0 subscript denoting the classical field is omitted here. With $\alpha = g^{\frac{1}{2}}$ and $\partial_m \alpha = \frac{1}{2} g^{-\frac{1}{2}} \partial_m(-v^2)$. In eq. (2.10)

$$g^{-\frac{1}{2}} g^{mn} \partial_m g^{\frac{1}{2}} \partial_n g^{\frac{1}{2}} - \frac{1}{2} \partial_n \left(g^{-\frac{1}{2}} g^{mn} \partial_m \left(g^{\frac{1}{2}}\right)^2\right)$$ (B.11)

$$= \frac{3}{4} g^{-\frac{3}{2}} \partial v^2 \cdot \partial v^2 + g^{-\frac{5}{2}}(v \cdot \partial v^2)^2 + \frac{1}{2} g^{-\frac{3}{2}} \partial \cdot v(v \cdot \partial v^2) + \frac{1}{2} g^{-\frac{1}{2}}(\partial^2 v^2 + g^{-1} v^m v^n \partial_{mn} v^2).$$

Noticing the identities,

$$\partial \cdot v = -\frac{1}{2} g^{-1} v \cdot \partial v^2 + g^{\frac{1}{2}} K,$$

$$v \cdot \partial(\partial \cdot v) = -\frac{1}{2} g^{-2}(v \cdot \partial v^2)^2 - \frac{1}{2} g^{-1} v \cdot \partial(v \cdot \partial v^2) + v \cdot \partial(g^{\frac{1}{2}} K),$$

$$\partial^2 v^2 = \partial_m \partial^m (v_n v^n) = \partial_m(v_n^m v^n + v_n v^{mn}) = 2\partial_m(v^{mn} v_n) = 2v \cdot \partial(\partial \cdot v) + 2v^{mn} v_{mn},$$

$$v^m v^n \partial_{mn} v^2 = v_m^m[(\partial_n v^2) v^n] - v^m \partial_n v^2 \partial_m v^n = v^m \partial_m[(\partial_n v^2) v^n] - \partial_n v^2 v^m \partial^n v_m$$

$$= v \cdot \partial(v \cdot \partial v^2) - \frac{1}{2} \partial v^2 \cdot \partial v^2,$$ (B.12)

where the first and the second equations are just K expression, eq. (B.12) reduces to

$$
\begin{aligned}
g^{-\frac{1}{2}} g^{mn} & \partial_m g^{\frac{1}{2}} \partial_n g^{\frac{1}{2}} - \frac{1}{2} \partial_n \left(g^{-\frac{1}{2}} g^{mn} \partial_m \left(g^{\frac{1}{2}} \right)^2 \right) \\
&= \frac{3}{4} g^{-\frac{3}{2}} \partial v^2 \cdot \partial v^2 + g^{-\frac{5}{2}} (v \cdot \partial v^2)^2 - \frac{1}{4} g^{-\frac{5}{2}} (v \cdot \partial v^2)^2 + \frac{1}{2} g^{-\frac{1}{2}} [2 v \cdot \partial (\partial \cdot v) + 2 v^{mn} v_{mn}] \\
&\quad + \frac{1}{2} g^{-\frac{3}{2}} \left[v \cdot \partial (v \cdot \partial v^2) - \frac{1}{2} \partial v^2 \cdot \partial v^2 \right] + \frac{1}{2} g^{-1} K (v \cdot \partial v^2) \\
&= g^{\frac{1}{2}} \left[\frac{1}{g} v^{mn} v_{mn} + \frac{1}{2g^2} \partial v^2 \cdot \partial v^2 + \frac{1}{4g^3} (v \cdot \partial v^2)^2 \right] + \frac{1}{2} g^{-1} K (v \cdot \partial v^2) + g^{-\frac{1}{2}} v \cdot \partial (g^{\frac{1}{2}} K) \\
&= g^{\frac{1}{2}} (K_b^a K_a^b) + \frac{1}{2} g^{-1} K (v \cdot \partial v^2) + g^{-\frac{1}{2}} g^{-\frac{1}{2}} \frac{1}{2} v \cdot \partial (-v^2) K + g^{-\frac{1}{2}} g^{\frac{1}{2}} v \cdot \partial K \\
&= g^{\frac{1}{2}} (K_b^a K_a^b) + v \cdot \partial K,
\end{aligned} \tag{B.13}
$$

where eq. (B.11) has been used to convert the expression to a compact form. When the equation of motion $K = 0$ is applied, $R = K_b^a K_a^b - K^2 = K_b^a K_a^b$ and $v \cdot \partial K = 0$. Finally, eq. (B.13) becomes

$$
g^{-\frac{1}{2}} g^{mn} \partial_m g^{\frac{1}{2}} \partial_n g^{\frac{1}{2}} - \frac{1}{2} \partial_n \left(g^{-\frac{1}{2}} g^{mn} \partial_m (g^{\frac{1}{2}})^2 \right) = g^{\frac{1}{2}} R. \tag{B.14}
$$

Acknowledgments

References

[1] A.M. Polyakov, *Quantum geometry of bosonic strings*, *Phys. Lett.* **B 103** (1981) 207.

[2] T.E. Clark, M. Nitta and T. ter Veldhuis, *Brane dynamics from non-linear realizations*, *Phys. Rev.* **D 67** (2003) 085026 [hep-th/0208184].

[3] T.E. Clark, S.T. Love, M. Nitta, T. ter Veldhuis and C. Xiong, *Oscillating p-branes*, *Phys. Rev.* **D 76** (2007) 105014 [hep-th/0703179].

[4] E.S. Fradkin and A.A. Tseytlin, *On quantized string models*, *Ann. Phys. (NY)* **143** (1982) 413.

[5] P.A.M. Dirac, *An extensible model of the electron*, *Proc. Roy. Soc. Lond.* **A 268** (1962) 57;
Y. Nambu, *Strings, monopoles and gauge fields*, *Phys. Rev.* **D 10** (1974) 4262;
T. Goto, *Relativistic quantum mechanics of one-dimensional mechanical continuum and subsidiary condition of dual resonance model*, *Prog. Theor. Phys.* **46** (1971) 1560.

[6] M. Caselle, M. Hasenbusch and M. Panero, *String effects in the 3D gauge Ising model*, *JHEP* **01** (2003) 057 [hep-lat/0211012]; *On the effective string spectrum of the tridimensional Z_2 gauge model*, *JHEP* **01** (2006) 076 [hep-lat/0510107];
M. Panero, *A numerical study of confinement in compact QED*, *JHEP* **05** (2005) 066 [hep-lat/0503024].

[7] J. Honerkamp, *Chiral multiloops*, *Nucl. Phys.* **B 36** (1972) 130.

[8] D.G. Boulware and L.S. Brown, *Symmetric space scalar field theory*, *Ann. Phys. (NY)* **138** (1982) 392.

[9] K. Fujikawa, *Comment on chiral and conformal anomalies*, *Phys. Rev. Lett.* **44** (1980) 1733.

[10] N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, Cambridge U.K. (1982).

[11] G. 't Hooft and M.J.G. Veltman, *One loop divergencies in the theory of gravitation*, *Annales Poincaré Phys. Theor.* **A20** (1974) 69.